Linear ODEs: an algebraic perspective

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The purpose of this mini-course is twofold. On one hand it aims to introduce and advertise a natural, flexible and elegant purely–algebraic approach to the well-known classical theory of linear ODEs with constant coefficients, and to introduce generalised Wronskians associated to a fundamental system of solutions. Elementary applications will be shown, e.g. to the computation of the exponential of a square matrix without reducing to Jordan canonical form. On the other hand it wishes to bring to the fore a number of relationships with other branches of mathematics. Examples include the theory of symmetric functions, the theory of the universal decomposition algebras associated to a polynomial, derivations of the exterior algebra of a free module, D-modules, Schubert calculus for the complex Grassmannian, boson-fermion correspondence in the representation theory of infinite–dimensional Lie algebras (like the Virasoro algebra) seen as an infinite–dimensional analogue of Poincaré's duality for the complex Grassmannians.

The schedule may vary according to the taste and/or the composition of the audience. The level of the course will be elementary, given that more than seventy percent of the material can be understood with a basic knowledge of polynomial algebras and of the Leibniz rule for the product of two differentiable functions.

The tentative plan is:

1. Overview of the course, contents and aims. Universal solutions to the universal linear ODEs with constant coefficients; universal Cauchy theorem on the existence and uniqueness for linear ODEs; universal Euler formula, generalising $e^{it} = \cos t + \sqrt{-1}\sin t$. Formal Laplace transform and its properties. Generalised Wronskians associated to the kernel of the universal differential operator.

2. The Wronskian and its derivatives. Application to the computation of the exponential of a square matrix with entries in an arbitrary $\mathbb{Q}$-algebra. A generalization of $\cos^2 t + \sin^2 t = 1$. Nice combinatorial features enjoyed by the generalised Wronskians: hook-length formula for the the nth derivative of a Wronskian; Jacobi-Trudy formula for generalised Wronskians; connection with Schubert calculus on complex grassmannians.

3. A glimpse onto exterior powers, exterior algebra of a free module and its derivations. A parallel with generalised Wronskians. The Jacobi-Trudy formula for Wronskians phrased in the language of exterior algebra derivations.

4. Linear ODEs (with constant coefficients) of infinite order. Semi-infinite exterior power of the module generated by a basis of solutions to a linear ODEs of infinite order. Fock spaces. The boson-fermion correspondence in

the theory of representations of infinite dimensional Lie algebras, phrased in terms of derivations on the semi-infinite exterior power of the module of solutions to a linear ODEs of infinite order.

5. Discussing earlier topics that deserve a more detailed study, or, if time permits, to mention further connections to other subjects, e.g. to the Hirota-Plücker bilinear equations in the theory of the KP (Kadomtsev–Petviashvili) hierarchy.

References

[1] E. Arbarello, *Sketches of KdV*, Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), Contemp. Math., **312**, Amer. Math. Soc., Providence, RI, 2002, 9–69.

[2] S. C. Coutinho, *A primer of algebraic D-modules*, London Mathematical Society Student Texts, **33**, Cambridge University Press, Cambridge, 1995.

[3] L. Gatto, *Schubert Calculus via Hasse–Schmidt Derivations*, Asian J. Math. **9**, No. 3, 315–322, (2005).

[4] L. Gatto, *Schubert Calculus: An Algebraic Introduction*, Publicações Matemáticas do IMPA, 25º Colóquio Brasileiro de Matemática, Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2005.

[5] L. Gatto, *The Wronskian and its Derivatives*, Atti Acc. Peloritana dei Pericolanti, Physical, Mathematical, and Natural Sciences, **89**, no. 2 (2011), 14 pages.

[6] L. Gatto, I. Scherbak, *Linear ODEs, Wronskians and Schubert Calculus*, Moscow Math. J., 2012 (in print).

[7] L. Gatto, I. Scherbak, *On Generalized Wronskians*, in "Contributions to Algebraic Geometry. Impanga Lecture Notes", P. Pragacz ed., EMS Series of Congress Reports, 2012 (in print).

[8] Huang Yong-nian, *The Explicit Solution of Homogeneous Linear Ordinary Differential Equations with Constant Coefficients*, Applied Mathematics and Mechanics (English Edition, VoI. **13**, No.12, Dec. 1992)

[9] V. G. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. **10**, AMS, 1996.

[10] V. G. Kac, A. K. Raina, *Highest Weight Representations of Infinite Dimensional Lie Algebras*, Advanced Studies in Mathematical Physics, Vol. **2**, World Scientific, 1987.

[11] E. Liz, *A Note on the Matrix Exponential*, SIAM Rev., Vol. **40**, No. 3, pp. 700–702, 1998.

[12] M. E. Kazarian, S. K. Lando, *An algebro-geometric proof of Witten's conjecture*, J. Amer. Math. Soc. **20** (2007), no. 4, 1079–1089.

[13] I. H. Leonard, *The Matrix Exponential*, SIAM rev., Vol. **38**, No. 3, pp. 507–512, 1996.

[14] M. Mulase, *Algebraic theory of the KP equations*, Perspectives in Mathematical Physics, Conf. Proc. Lecture Notes Math. Phys., **III**, Int. Press, Cambridge, MA, 151–217, 1994.

[15] E. J. Putzer, *Avoiding the Jordan canonical form in the discussion of linear systems with constant coefficients*, Amer. Math. Monthly **73**, 1966, 2–7.